

A note on the distribution of normalized prime gaps

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1 Introduction

The Prime Number Theorem implies that the average value of

$$(1.1) \quad d_n = p_{n+1} - p_n$$

is $(1 + o(1)) \log p_n$ if $n \in [N, 2N]$, for example, where $\mathbb{P} = \{p_i\}_{i=1}^\infty$ is the set of primes. This motivates the investigation of the sequence $\{d_n / \log p_n\}_{n=1}^\infty$ or $\{d_n / \log n\}_{n=1}^\infty$ (which is asymptotically equal). Erdős formulated the conjecture that the set of its limit points

$$(1.2) \quad J = \left\{ \frac{d_n}{\log n} \right\}' = [0, \infty].$$

He writes in [Erd 1955]: “It seems certain that $d_n / \log n$ is everywhere dense in $(0, \infty)$ ” (after mentioning the conjecture $\liminf_{n \rightarrow \infty} d_n / \log n = 0$). The fact that $\infty \in J$ was proved already in 1931 by Westzynthius [Wes 1931].

In 2005 Goldston, Yıldırım and the author [GPY 2006], [GPY 2009] showed $0 \in J$ which is the hitherto only concrete known element of J . On the other hand already 60 years ago Ricci [Ric 1954] and Erdős [Erd 1955] proved (simultaneously and independently) that J has a positive Lebesgue measure. A partial result towards the full conjecture (1.2) was shown by the author in [Pin 2013arX] according to which there exists an ineffective constant c such that

$$(1.3) \quad [0, c] \subset J.$$

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In a recent work W. Banks, T. Freiberg and J. Maynard [BFM 2014arX] proved that for any sequence of $k = 9$ nonnegative real numbers $\beta_1 \leq \beta_2 \leq \dots \leq \beta_k$ we have

$$(1.4) \quad \{\beta_j - \beta_i : 1 \leq i < j \leq k\} \cap J \neq \emptyset.$$

As a corollary they obtained that if λ denotes the Lebesgue measure, then

$$(1.5) \quad \lambda([0, T] \cap J) \geq (1 + o(1))T/8.$$

2 Generalization and Improvement

The purpose of this note is to generalize this result for the case when d_n is normalized by a rather general function $f(n) \rightarrow \infty$, that is to consider instead of J the more general case of the set of limit points

$$(2.1) \quad J_f = \left\{ \frac{d_n}{f(n)} \right\}'$$

where we require from f to belong to the class \mathcal{F} below.

Definition. A function $f(n) \nearrow \infty$ belongs to \mathcal{F} if for any $\varepsilon > 0$

$$(2.2) \quad (1 - \varepsilon)f(N) \leq f(n) \leq (1 + \varepsilon)f(N) \quad \text{for } n \in [N, 2N], \quad N > N_0,$$

further if

$$(2.3) \quad f(n) \ll \log n \log_2 n \log_4 n / (\log_3 n)^2$$

where $\log_\nu n$ denotes the ν -times iterated logarithm.

The first condition means that $f(n)$ is slowly oscillating, while the second one that it does not grow more quickly than the Erdős–Rankin function, which until the recent dramatic new developments by Maynard [May 2014arX], Ford–Green–Konyagin–Tao [FGKT 2014arX], and Ford–Green–Konyagin–Maynard–Tao [FGKMT 2014arX] described the largest known gap between consecutive primes. The improvement means that it is sufficient to work with $k = 5$ values of β_i in (1.4) instead of $k = 9$ values. As an immediate corollary we obtain a lower bound $(1 + o(1))T/4$ instead of (1.5) for the Lebesgue measure of the more general set $[0, T] \cap J_f$.

Theorem 1. *If $f \in \mathcal{F}$, then for any sequence of $k = 5$ nonnegative real numbers $\beta_1 \leq \beta_2 \leq \dots \leq \beta_k$ we have*

$$(2.4) \quad \{\beta_j - \beta_i : 1 \leq i < j \leq k\} \cap J_f \neq \emptyset.$$

Corollary 2. *If $f \in \mathcal{F}$, then*

$$(2.5) \quad \lambda([0, T] \cap J) \geq (1 + o(1))T/4.$$

In an earlier work [Pin 2013arX] we showed that for any $f \in \mathcal{F}$ there exists an ineffective constant c_f such that $[0, c_f] \subset J_f$. We further remark that since β_i can be arbitrarily large, Theorem 1 includes the improvement of the Erdős–Rankin function given in (2.3) proved recently in [May 2014arX] and [FGKT 2014arX]. (We note that the proof uses some refinement of the argument of [May 2014arX], so it does not represent an independent new proof.)

In connection with the original Erdős conjecture for general $f \in \mathcal{F}$ we remark that it was proved in [Pin 2014arX] that the conjecture is in some sense valid for almost all functions $f \in \mathcal{F}$. More precisely it was shown in [Pin 2014arX] that if $\{f_n\}_1^\infty \in \mathcal{F}$ with $\lim_{x \rightarrow \infty} f_{n+1}(x)/f_n(x) = \infty$, then

$$(2.6) \quad J_{f_n} = [0, \infty]$$

apart from at most 98 exceptional functions f_n .

3 Proof

The generalization for the case $f \in \mathcal{F}$ instead of the single case $f = \log n$ runs completely analogously to the proofs in [Pin 2014arX] so we will only describe how to improve $k = 9$ to $k = 5$ in Theorem 1 which leads to the improved Corollary 2 in the same simple way as described in the Introduction of the work of Banks, Freiberg and Maynard [BFM 2014arX].

The result will follow from the following improvement of Theorem 4.3 of their work. Let \mathcal{Z} be given by (4.8) of [BFM 2014arX].

Theorem 3. *Let m, k and $\varepsilon = \varepsilon(k)$ be fixed. If k is a sufficiently large multiple of $4m + 1$ and ε is sufficiently small, there is some $N(m, k, \varepsilon)$ such that the following holds for $N \geq N(m, k, \varepsilon)$ with*

$$(3.1) \quad w = \varepsilon \log N, \quad W = \prod_{p \leq w, p \nmid \mathcal{Z}} p.$$

Let $\mathcal{H} = \{h_1, \dots, h_k\}$ be an admissible k -tuple (that is it does not cover all residue classes mod p for any prime p) such that

$$(3.2) \quad 0 \leq h_1 < \dots < h_k \leq N$$

and

$$(3.3) \quad p \mid \prod_{1 \leq i < j \leq k} (h_j - h_i) \implies p \leq w.$$

Let $\mathcal{H} = \mathcal{H}_1 \cup \dots \cup \mathcal{H}_{4m+1}$ be a partition of \mathcal{H} into $4m+1$ sets of equal size and let b be an integer with

$$(3.4) \quad \left(\prod_{i=1}^k (b + h_i), W \right) = 1.$$

Then there is some $n \in (N, 2N]$ with $n \equiv b \pmod{W}$ and some set of distinct indices $\{i_1, \dots, i_{m+1}\} \subseteq \{1, \dots, 4m+1\}$ such that

$$(3.5) \quad |\mathcal{H}_i(n) \cap \mathbb{P}| \geq 1 \quad \text{for all } i \in \{i_1, \dots, i_{m+1}\}.$$

Remark. The original analogous statement (4.20) of [BFM 2014arX] should have been stated with ≥ 1 instead of $= 1$ (oral communication of James Maynard). This form is enough to imply their Corollary 1.2 or our Corollary 2.

The needed change in the Deduction of Theorem 4.3 is the following. First, using $4m+1 \mid k$ we write

$$(3.6) \quad \mathcal{H} = \mathcal{H}_1 \cup \dots \cup \mathcal{H}_{4m+1}$$

as a partition of \mathcal{H} into $4m+1$ sets each of size $k/(4m+1)$. Instead of the quantity S in [BFM 2014arX] we introduce with a new parameter $\alpha = \alpha(m)$ the new quantity $S(\alpha)$, where α will be chosen relatively small (we will see that $\alpha(m) = 1/(5m)$ is a good choice, for example). Thus, let with a further parameter β

$$(3.7) \quad S(\alpha, \beta) = \sum_{N < n \leq 2N} \left(\sum_{i=1}^k 1_{\mathbb{P}}(n + h_i) - \beta m - \alpha \sum_{j=1}^{4m+1} \sum_{\substack{h, h' \in \mathcal{H}_j \\ h \neq h'_2}} 1_{\mathbb{P}}(n + h) 1_{\mathbb{P}}(n + h') \right) \\ \times \left(\sum_{\substack{d_1, \dots, d_k \\ d_i \mid n + h_i \forall i}} \lambda_{d_1, \dots, d_k} \right)^2$$

where under the summation sign we consider unordered pairs $h, h' \in \mathcal{H}_j$. Let

$$(3.8) \quad \beta = \beta(\alpha) = \max_{\ell \in \mathbb{Z}^+} \left(\ell - \alpha \binom{\ell}{2} \right).$$

Then the contribution of any set \mathcal{H}_j to $S(\alpha, \beta) = S(\alpha)$ is at most β , so if we have for every $n \in (N, 2N]$ at most m sets of the form \mathcal{H}_j with

$$(3.9) \quad \sum_{h \in \mathcal{H}_j} 1_{\mathbb{P}}(n + h) > 0,$$

then consequently

$$(3.10) \quad S(\alpha) \leq 0.$$

In contrary to the choice $\varrho \in (0, 1)$ and $\delta \varrho \log k = 2m$ of [BFM 2014arX] we will choose now $\delta \varrho \log k$ much larger

$$(3.11) \quad \delta \varrho \log k = u := \frac{4m+1}{4\alpha}, \quad \alpha = \frac{1}{5m}, \quad \varrho \in (0, 1).$$

This implies with an easy calculation

$$(3.12) \quad \beta = \frac{5m+1}{2}.$$

Using the same argument for the estimation of the negative double sum as [BFM 2014arX] we obtain a choice of a function F such that

$$(3.13) \quad \begin{aligned} S(\alpha) &= \frac{N}{W} B^{-k} I_k(F) \left(\sum_{i=1}^k \frac{u}{k} (1 + O(\gamma)) - \beta m - 4\alpha \sum_{j=1}^{4m+1} \sum_{\substack{h, h' \in \mathcal{H}_j \\ h \neq h'}} \frac{u^2}{k^2} (1 + O(\delta + \gamma)) \right) \\ &= \frac{N}{W} B^{-k} I_k(F) \left(u(1 + O(\gamma)) - \frac{(5m+1)m}{2} - \frac{4(4m+1)}{5m} \binom{k/(4m+1)}{2} \frac{u^2}{k^2} (1 + O(\delta + \gamma)) \right). \end{aligned}$$

By the above choice of the parameters in (3.11) we have from (3.13) with $\gamma = (\log k)^{-1/2}$

$$(3.14) \quad \begin{aligned} \frac{S(\alpha) W B^k}{N I_k(F)} &\geq \frac{5m(4m+1)(1 + O(\gamma))}{4} - \frac{(5m+1)m}{2} \\ &\quad - \frac{5m(4m+1)(1 + O(\delta + \gamma))}{8} \\ &= \frac{m(1 + O(m(\delta + \gamma)))}{8} > 0, \end{aligned}$$

which contradicts to (3.10).

In order to see the validity of the last inequality we can choose

$$(3.15) \quad m < (\log k)^{1/4} \iff \delta \asymp \left(\frac{m^2}{\log k} \right)$$

which implies $m\gamma = o(1)$ and $m\delta = o(1)$. This contradiction proves our Theorem 1. Corollary 2 follows from it in the same way as Corollary 1.2 from Theorem 1.1 in [BFM 2014arX].

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